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Renormalization of the Topological Charge in Yang–Mills Theory

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Abstract

The conditions leading to a nontrivial renormalization of the topological charge in four–dimensional Yang–Mills theory are discussed. It is shown that if the topological term is regarded as the limit of a certain nontopological interaction, quantum effects due to the gauge bosons lead to a finite multiplicative renormalization of the θ–parameter while fermions give rise to an additional shift of θ. A truncated form of an exact renormalization group equation is used to study the scale dependence of the θ–parameter. Possible implications for the strong CP–problem of QCD are discussed.

1 Introduction

One of the most interesting aspects of Yang–Mills theories in 4 spacetime dimensions is the possibility of adding a term $S_{\text{top}} = i\theta Q$ to their action which is proportional to the topological charge Q :

$$S_{\text{top}}[A] \equiv i\theta \frac{\bar{g}^2}{32\pi^2} \int d^4x F^a_{\mu\nu} {}^*F^a_{\mu\nu} \tag{1.1}$$

From a hamiltonian point of view the vacuum angle θ can be regarded as a kind of quasi–momentum which owes its existence to the periodic structure of the Yang–Mills vacuum and which is similar to the quasi–momentum of Bloch waves in periodic potentials. For this reason it was commonly believed that θ is not renormalized by radiative corrections, and that all observables are 2π–periodic in θ. It came as a surprise therefore that explicit one–loop calculations [1, 2] within standard lagrangian perturbation theory revealed a finite renormalization of the topological charge. Later on it was observed [3] that the zero–modes of the inverse gluon propagator also lead to a renormalization of the topological charge and that their contribution cancels precisely the finite renormalization found earlier [1, 2]. Even though there seems to be no net renormalization left the cancellation which leads to this result is of a rather delicate nature. The first one of the two contributions has the character of a triangle anomaly and originates in the ultraviolet while the second one, due to the zero–modes, is a typical infrared effect. As the cancellation has been established at the one–loop level only one might wonder if it persists at higher orders of perturbation theory and at the non–perturbative level. Since the infrared behavior of QCD–type theories is only very poorly understood one cannot exclude the possibility that the actual contribution of the zero–modes differs from the lowest order result and that the compensation is incomplete therefore.

In this paper we shall explain in which sense one may talk about a renormalization of the topological charge or of the θ–parameter, and how this can be reconciled with the hamiltonian non–renormalization argument. Both the infrared and the ultraviolet effects will be investigated in detail, and we shall see that generically there is no perfect compensation among them. Because we are aiming at a clean separation of the relevant momentum scales, we employ the method of the exact renormalization group equations [4]. The basic idea is to consider

a scale–dependent effective action Γ_k, henceforth referred to as the "effective average action", which obtains from the classical action S by integrating out only the field modes with momenta larger than the infrared cutoff k. The conventional effective action Γ is recovered in the limit $k \to 0$, i.e., in the space of all actions, the renormalization group trajectory Γ_k, $0 \le k < \infty$, interpolates between the classical action $S = \Gamma_{k\to\infty}$ and the standard effective action $\Gamma = \Gamma_{k\to 0}$.

In ref.[5] we introduced an exact evolution equation for gauge theories which maintains gauge invariance at all intermediate scales.[1] For a pure Yang–Mills theory it reads

$$
\begin{aligned}
k\frac{d}{dk}\Gamma_k[A,\bar{A}] \;=\; & \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[A,\bar{A}] + R_k\left(\Delta[\bar{A}]\right)\right)^{-1} k\frac{d}{dk}R_k\left(\Delta[\bar{A}]\right)\right] \\
& - \mathrm{Tr}\left[\left(-D_\mu[A]D_\mu[\bar{A}] + R_k\left(-D^2[\bar{A}]\right)\right)^{-1} k\frac{d}{dk}R_k\left(-D^2[\bar{A}]\right)\right]
\end{aligned}
\tag{1.2}
$$

As we use the background gauge fixing technique [9], the functional Γ_k depends both on the usual classical average field A^a_μ and on the background field $\bar{A}^a_\mu$. The equation (1.2) has to be solved for the initial condition

$$
\Gamma_\infty[A,\bar{A}] = S[A] + \frac{1}{2\alpha}\int d^4x\left(D^{ab}_\mu[\bar{A}](A^b_\mu - \bar{A}^b_\mu)\right)^2 \tag{1.3}
$$

Apart from the classical action $S[A]$, Γ_∞ also contains the well–known background gauge fixing term [9]. $\Gamma_k^{(2)}[A,\bar{A}]$ denotes the matrix of second functional derivatives of Γ_k with respect to A, at fixed $\bar{A}$. The function R_k describes the precise form of the infrared cutoff. It is arbitrary to a large extent, but it has to satisfy $\lim\limits_{u\to\infty} R_k(u) = 0$ and $\lim\limits_{u\to 0} R_k(u) = Z_k k^2$ for some constant Z_k (see below). Usually we shall use the parametrization

$$
R_k(\Delta) = Z_k k^2 R^{(0)}\left(\frac{\Delta}{Z_k k^2}\right) \tag{1.4}
$$

with $R^{(0)}$ smoothly interpolating between $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$. The operator Δ is used to distinguish, in a gauge invariant way, "high momentum" modes from "low momentum" modes. Expanding all field modes in terms of eigenfunctions of Δ, only the modes with eigenvalues $p^2 > k^2$ are integrated out. In practice Δ consists essentially of (minus) the covariant Laplacian $-D^2[\bar{A}]$ with

[1] For alternative approaches see [6, 7, 8].

the covariant derivatives in the adjoint representation. The meaning of the factor Z_k in (1.4) is as follows. Assume that, at scale k, a certain field mode has a massless inverse propagator $Z'_k p^2$. Then we should use $Z_k \equiv Z'_k$ in (1.4) because this guarantees that the inverse propagator and the cutoff combine to $Z_k(p^2+k^2)$ for small eigenvalues p^2 of Δ. Hence the mode is cut off at $p^2 \approx k^2$ by a kind of field–dependent mass term. Actually Z_k may be chosen differently for different types of fields. It also may depend on $\bar{A}$, but not on A. (See [5, 10] for further details and [11] for a review of this approach.)

Approximate but still nonperturbative solutions of eq. (1.2) with (1.3) can be obtained by truncating the space of all action functionals. If one makes an ansatz for Γ_k with finitely many k–dependent parameters (generalized couplings) multiplying the field monomials which were retained, then the functional evolution equation becomes a set of ordinary differential equations for the generalized couplings. To be precise, in writing down eq. (1.2) we already made a special kind of truncation. As it stands the evolution equation neglects renormalization effects in the gauge fixing and the ghost sector.[2] In the cases studied so far this has led to rather reliable results [5, 12, 13].

The objective of this paper is to solve the renormalization group equation with the initial condition

$$S[A] = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu} + i\theta_{\rm bare}\frac{\bar{g}^2}{32\pi^2}\int d^4x F^a_{\mu\nu}\,{}^*\!F^a_{\mu\nu} \tag{1.5}$$

We are going to allow for a scale-dependent θ–parameter, $\theta \equiv \theta(k)$, and we shall follow its evolution from the bare value $\theta_{\rm bare} \equiv \theta(\infty)$ down to the renormalized one, $\theta_{\rm ren} \equiv \theta(0)$.[3] The classical action (1.5) coincides with the effective action Γ_Λ at the UV cutoff $\Lambda \to \infty$. Let us see what happens if we lower the scale k from Λ to an infinitesimally lower scale $\Lambda - \delta k$. Near $k = \Lambda$ the Hessian $\Gamma_k^{(2)}$ which appears on the RHS of (1.2) is simply $\Gamma_\Lambda^{(2)} = S^{(2)}$. However, as a consequence of the topological nature of the θ–term, its matrix of second functional derivatives vanishes identically, and $S^{(2)}$ receives contributions only from the standard kinetic term $\frac{1}{4}F^2$. Therefore $\Gamma_\Lambda^{(2)}$ contains no parity-odd piece. This entails that the traces

[2] A detailed discussion of this approximation and the general form of the evolution equation can be found in refs. [10, 11].

[3] For a different evolution equation in the framework of the dilute instanton gas approximation see also ref.[14].

in (1.2) cannot produce a term proportional to the pseudoscalar $F^a_{\mu\nu}{}^*F^a_{\mu\nu}$ which could match a term

$$k\frac{d\theta(k)}{dk}\int d^4x F^a_{\mu\nu}{}^*F^a_{\mu\nu} \tag{1.6}$$

on the LHS of the equation. Hence $\frac{d\theta}{dk}=0$ at $k=\Lambda$, and $\theta(\Lambda-\delta k)=\theta(\Lambda)$ remains unchanged. Though the parity–even terms in Γ_k have changed while going from Λ to $\Lambda-\delta k$, we can repeat the above argument for the full range of scales between $k=\Lambda$ and $k=0$. The result is that $\theta(k)$ keeps its bare value $\theta(\Lambda)$ at all lower scales, i.e., it does not get renormalized.[4]

Within the renormalization group formalism, the above argument is the analog of the hamiltonian reasoning which leads to the conclusion that θ is not renormalized. The crucial question is how this can be reconciled with the explicit diagrammatic calculations in ref.[1] which yield a finite renormalization of the topological charge or, equivalently, of the θ–parameter. In our framework this phenomenon can be explained as follows. Let us temporarily replace the topological term in (1.5) by

$$S_\theta[A,\phi]=i\theta_{\rm bare}\frac{\bar g^2}{32\pi^2}\int d^4x\,\phi(x)F^a_{\mu\nu}{}^*F^a_{\mu\nu} \tag{1.7}$$

Here $\phi(x)$ is a localized external pseudoscalar field which we shall not quantize. We interpret the term (1.7) as the coupling of a pseudoscalar "meson" $\phi(x)$ to the gluon field with a bare coupling strength $\theta_{\rm bare}=\theta(\Lambda)$. If we now ask how the coupling $\theta=\theta(k)$ depends on the scale k we indeed will get a nontrivial answer. The second variation of (1.7) is no longer zero, but rather proportional to $\int d^4x\partial_\mu\phi K^{(2)}_\mu$ where K_μ is the Chern–Simons current. Therefore the k–evolution produces all sorts of terms involving both ϕ and A^a_μ. Among them there is the term $\theta(k)\int d^4x\phi F^a_{\mu\nu}{}^*F^a_{\mu\nu}$ with a scale dependent coupling $\theta(k)$. After having solved the evolution equation for the renormalization group trajectory $\Gamma_k[A,\bar A;\phi]$ we can ask what happens if we allow $\phi(x)$ to approach unity for all x. Then, on the one hand, (1.7) is the original topological term again, but on the other hand also the running interaction term $\theta(k)\int F^a_{\mu\nu}{}^*F^a_{\mu\nu}$ becomes proportional to the topological charge *but with a renormalized prefactor* $\theta(k)$. Later on we shall demonstrate that – if understood in this sense – a renormalization of the topological charge is indeed possible.

[4] For the general form of the evolution equation [10] this is still true.

The situation is most concisely described by saying that the k–evolution and the limit $\phi(x) \to 1$ do not commute. If one sets $\phi(x) \equiv 1$ from the outset the topological charge is not renormalized in accordance with the hamiltonian arguments. If one considers the topological term as the limit of the interaction term $\int \phi F^a_{\mu\nu} {}^*\!F^a_{\mu\nu}$ for the slowly varying ϕ but lets $\phi(x) \to 1$ only at the end of all calculations then one finds a nontrivial renormalization of θ. Clearly the two different procedures correspond to different physical situations; which is the correct one cannot be decided on purely formal grounds. In the following we shall study the second option throughout.

The remaining sections of the paper are organized as follows. In section 2 we derive and solve the evolution equation of $\theta(k)$ for all non–zero values of k. We establish that $\theta(k)$ has a finite discontinuity at $k \to \infty$ and is constant otherwise. In section 3 we investigate the limit $k \to 0$ and demonstrate that $\theta(k)$ has a second discontinuity at $k = 0$. In section 4 we summarize our results and comment on possible applications in the context of the strong CP problem of QCD. In the main part of this paper we discuss the more interesting effects due to the quantized gauge field. In the appendix we include fermion loops, and the reader should compare the respective calculations for gauge bosons and fermions.

2 Ultraviolet Renormalization

We consider pure Yang–Mills theory with an arbitrary (semisimple, compact) gauge group G in 4–dimensional euclidean space. In order to solve the evolution equation we make an ansatz of the following form[5]

$$
\begin{aligned}
\Gamma_k[A,\bar{A};\phi] \;&=\; Z_F(k)\int d^4x\left\{\frac{1}{4}F^a_{\mu\nu}(A)F^a_{\mu\nu}(A)+\frac{1}{2\alpha}\left(D^{ab}_\mu[\bar{A}](A^b_\mu-\bar{A}^b_\mu)\right)^2\right\}\\
&+\; i\theta(k)\frac{\bar{g}^2}{32\pi^2}\int d^4x\,\phi(x)\,F^a_{\mu\nu}(A)\,{}^*\!F^a_{\mu\nu}\,(A)
\end{aligned}
\tag{2.1}
$$

It satisfies the initial condition (1.3) with the classical action (1.5) for the values $Z_F(\infty)=1$ and $\theta(\infty)=\theta_{\rm bare}$. The ansatz (2.1) truncates the space of all actions to a 2–dimensional subspace parametrized by Z_F and θ. If one inserts (2.1) into the evolution equation (1.2) one obtains ordinary differential equations for the funtions $Z_F(k)$ and $\theta(k)$. In order to fully specify the evolution equation one has to make a choice for the cutoff operator $\Delta[\bar{A}]$. As in refs.[5, 10] we take

$$
\Delta[\bar{A}]=\Gamma^{(2)}_k[A=\bar{A},\bar{A};\phi=0]
\tag{2.2}
$$

but at the level of physical quantities neither the precise definition of Δ nor that of $R^{(0)}$ will matter.

In the approximation used here corrections to the gauge fixing term are neglected, and therefore the background field $\bar{A}$ enters (2.1) only via the classical gauge fixing term. This means that it is sufficient for our purposes to know $\Gamma_k[A,\bar{A}=A;\phi]$ because from the F^2– and the $F{}^*\!F$–term we can read off $Z_F(k)$ and $\theta(k)$, respectively. For $\bar{A}=A$ the LHS of (1.2) reads

$$
\begin{aligned}
k\frac{d}{dk}\Gamma_k[A,A;\phi] \;&=\; \frac{1}{4}k\frac{dZ_F(k)}{dk}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}\\
&+\; ik\frac{d\theta(k)}{dk}\frac{\bar{g}^2}{32\pi^2}\int d^4x\,\phi(x)F^a_{\mu\nu}{}^*\!F^a_{\mu\nu}
\end{aligned}
\tag{2.3}
$$

From now on all field strengths and covariant derivatives are constructed from A. In evaluating the RHS of the evolution equation we have to recall that $\Gamma^{(2)}_k\equiv$

[5]We write $\bar{g}$ for the bare gauge coupling and $D^{ab}_\mu[A]=\partial_\mu\delta^{ab}-i\bar{g}A^c_\mu(T^c)^{ab}$ with $(T^c)^{ab}=-if^{cab}$ for the covariant derivative in the adjoint representation. Furthermore, ${}^*\!F^a_{\mu\nu}\equiv\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$ with $\varepsilon_{1234}=1$.

$\Gamma_k^{(2)}[A, \bar{A}; \phi]$ is the matrix of second derivatives with respect to A only; hence these derivatives have to be performed *before* one sets $\bar{A} = A$. Keeping this in mind one arrives at

$$\Gamma_k^{(2)}[A, A; \phi] = Z_F(k)\mathcal{D} + i\theta(k)\frac{\bar{g}^2}{8\pi^2}\mathcal{V} \tag{2.4}$$

with the operators

$$\begin{aligned} \mathcal{D}^{ab}_{\mu\nu} &= -D^{ac}_\alpha D^{cb}_\alpha \delta_{\mu\nu} + 2i\bar{g}F^{ab}_{\mu\nu} + (1-\alpha^{-1})D^{ac}_\mu D^{cb}_\nu \\ \mathcal{V}^{ab}_{\mu\nu} &= \varepsilon_{\mu\nu\alpha\beta}(\partial_\alpha \phi)D^{ab}_\beta \end{aligned} \tag{2.5}$$

Here $F^{ab}_{\mu\nu} \equiv F^c_{\mu\nu}(T^c)^{ab}$ is the field strength matrix in the adjoint representation. When one inserts (2.4) into (1.2) consistency requires us to retain only the first terms of the derivative expansion. Because $\Gamma_k[A, A; \phi]$ is a gauge invariant functional of A, the lowest order terms are proportional to $F^a_{\mu\nu}F^a_{\mu\nu}$ and $\phi F^a_{\mu\nu}{}^*\!F^a_{\mu\nu}$. These are the same field monomials as on the LHS, eq. (2.3), so we can compare their coefficients and deduce the differential equations for $Z_F(k)$ and $\theta(k)$. Expanding

$$\begin{aligned} (\Gamma_k^{(2)}[A, A; \phi] + R_k)^{-1} &= (Z_F(k)\mathcal{D} + R_k)^{-1} \\ -i\theta(k)\frac{\bar{g}^2}{8\pi^2}(Z_F(k)\mathcal{D} + R_k)^{-1} \;\; &\mathcal{V} \;\; (Z_F(k)\mathcal{D} + R_k)^{-1} + \mathcal{O}(\phi^2) \end{aligned} \tag{2.6}$$

the term independent of ϕ contains no ε–tensor and gives rise to the $F^a_{\mu\nu}F^a_{\mu\nu}$–invariant, whereas the term linear in ϕ contributes to $F^a_{\mu\nu}{}^*\!F^a_{\mu\nu}$. Thus we get the following decoupled equations

$$\begin{aligned} \frac{1}{4}k\frac{d}{dk}Z_F(k)\int d^4x F^a_{\mu\nu}F^a_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left[\Big(Z_F(k)\mathcal{D} + R_k(\Delta)\Big)^{-1} k\frac{d}{dk}R_k(\Delta)\right] \\ &\quad -\mathrm{Tr}\left[\Big(-D^2 + R_k(-D^2)\Big)^{-1} k\frac{d}{dk}R_k(-D^2)\right] + \cdots \end{aligned} \tag{2.7}$$

$$\begin{aligned} &k\frac{d}{dk}\theta(k)\int d^4x\, \phi(x) F^a_{\mu\nu}{}^*\!F^a_{\mu\nu} = \\ &-2\theta(k)\mathrm{Tr}\left[\Big(Z_F(k)\mathcal{D} + R_k(\Delta)\Big)^{-1}\mathcal{V}\Big(Z_F(k)\mathcal{D} + R_k(\Delta)\Big)^{-1} k\frac{d}{dk}R_k(\Delta)\right] + \cdots \end{aligned} \tag{2.8}$$

Our goal is to extract the pieces proportional to F^2 and $\phi F\tilde{F}$ from the traces (2.7) and (2.8). To do this we may insert any field configuration into the traces which discriminates unambiguously between the respective invariants. Because of the complicated operators involved, this procedure is by far more convenient than the standard derivative expansion techniques. We shall specify A_μ^a later on. For the time being let us only assume that for the gauge field chosen, the Yang–Mills equations $D_\mu^{ab}F_{\mu\nu}^b \equiv 0$ are satisfied. This has the following very useful consequence. If one defines the operators $\mathcal{D}_T$ and $D \otimes D$ by

$$\begin{aligned}(\mathcal{D}_T)_{\mu\nu}^{ab} &= \left(-D^2\delta_{\mu\nu} + 2i\bar{g}F_{\mu\nu}\right)^{ab} \\ (D \otimes D)_{\mu\nu}^{ab} &= D_\mu^{ac}D_\nu^{cb}\end{aligned} \tag{2.9}$$

then $[\mathcal{D}_T, D \otimes D] = 0$ for such fields. This implies that the operators

$$P_L = -\frac{D \otimes D}{\mathcal{D}_T} \qquad , \qquad P_T = 1 - P_L \tag{2.10}$$

are orthogonal projectors on generalized longitudinal and transverse gluon states in the background A: $P_{T,L}^2 = P_{T,L}$, $P_T P_L = 0$. The kinetic operator $\mathcal{D}$ decomposes according to

$$\mathcal{D} = \left[P_T + \frac{1}{\alpha}P_L\right]\mathcal{D}_T \tag{2.11}$$

Leaving the $\mathcal{V}$–term aside, the inverse propagator for the transverse and the longitudinal modes is $Z_F(k)\mathcal{D}_T$ and $\alpha^{-1}Z_F(k)\mathcal{D}_T$, respectively. Moreover, after setting $\bar{A} = A$, the cutoff operator (2.2) becomes $\Delta = Z_F(k)\mathcal{D}$. In view of the comments following eq.(1.4) this suggests the following choice for the factors Z_k entering the cutoff function R_k[6]:

$$Z_k = \left[P_T + \frac{1}{\alpha}P_L\right] Z_F(k) \tag{2.12}$$

Hence (1.4) becomes

$$R_k(\Delta) = \left\{1 - \left(1 - \alpha^{-1}\right)P_L\right\} Z_F(k)k^2 R^{(0)}\left(\mathcal{D}_T/k^2\right) \tag{2.13}$$

This form of R_k has to be used in the first trace on the RHS of (2.7) and in the one of (2.8), since these traces are due to the gauge boson fluctuations. The second

[6] A priori the Z_k's are defined in terms of $P_T[\bar{A}]$ and $P_L[\bar{A}]$.

trace in (2.7) stems from the Faddeev–Popov ghosts. Because the renormalization of their kinetic term is neglected, one simply sets $Z_k = 1$ there [5, 10].

The equation (2.7) for $Z_F(k)$ has been evaluated in ref.[5] already and we only quote the result here. The renormalized gauge coupling constant is defined by

$$g^2(k) = \bar{g}^2 Z_F(k)^{-1} \tag{2.14}$$

Its β–function is

$$\beta_{g^2} = k\frac{d}{dk}g^2(k) = g^2(k)\eta_F(k) \tag{2.15}$$

where

$$\eta_F(k) \equiv -k\frac{d}{dk}\ln Z_F(k) \tag{2.16}$$

denotes the anomalous dimension of the gauge field. Eq.(2.7) leads to the following β–function

$$\beta_{g^2} = -\frac{11T(G)}{24\pi^2}g^4\left[1-\frac{5T(G)}{24\pi^2}g^2\right]^{-1} \tag{2.17}$$

where $T(G)$ denotes the value of the quadratic Casimir operator in the adjoint representation: $f^{acd}f^{bcd} = T(G)\delta^{ab}$. With our conventions one has $T(G) = N$ for $G = SU(N)$. Eq.(2.17) is a nonperturbative result. It sums up contributions of all orders in g^2. Expanding for small g^2, the g^4–term coincides with the standard one–loop expression, and the g^6–term differs by only a few percent from the known 2–loop coefficient.

Turning now to the equation for $\theta(k)$, (2.8), the properties of P_L and P_T can be used to simplify it considerably:

$$k\frac{d}{dk}\theta(k)\int d^4x\phi(x)F^a_{\mu\nu}{}^*F^a_{\mu\nu} = 2\theta(k)Z_F(k)^{-1}\left\{T_1+T_2\right\} \tag{2.18}$$

with

$$\begin{aligned} T_1 &\equiv \mathrm{Tr}\left[\mathcal{V}\left\{1+(\alpha-1)P_L\right\}k\frac{d}{dk}\left(\mathcal{D}_T+k^2R^{(0)}(\mathcal{D}_T/k^2)\right)^{-1}\right] \\ T_2 &\equiv \eta_F(k)k^2\,\mathrm{Tr}\Big[\mathcal{V}\left\{1+(\alpha-1)P_L\right\}R^{(0)}(\mathcal{D}_T/k^2) \\ &\qquad\cdot\left(\mathcal{D}_T+k^2R^{(0)}(\mathcal{D}_T/k^2)\right)^{-2}\Big] \end{aligned} \tag{2.19}$$

The contribution T_1 is similar to what one encounters in a one–loop calculation with an IR cutoff, whereas the second term, T_2, contains the "renormalization

group improvement". The factor $\eta_F(k)$ arises when the k–derivative acts upon the factor Z_F contained in R_k.

Next we have to compute the coefficient of the $\phi F^a_{\mu\nu} {}^*\!F^a_{\mu\nu}$ –term contained in T_1 and T_2 as a function of k and as a functional of $R^{(0)}$. This is most easily done by assuming that A^a_μ has a covariantly–constant field strength, i.e., that $D^{ab}_\alpha F^b_{\mu\nu} = 0$, which implies $D^{ab}_\mu F^b_{\mu\nu} = 0$, of course. Though this does not mean that $F^a_{\mu\nu}$ is x–independent, the heat–kernel $K(s) = \exp(-s\mathcal{D}_T)$ in such backgrounds is known explicitly [15]. Only the first few terms of its expansion in powers of $F_{\mu\nu}$ can contribute to $F^a_{\mu\nu} {}^*\!F^a_{\mu\nu}$. They read

$$
\begin{aligned}
K^{ab}_{\mu\nu}(x,y;s) \; &= \; (4\pi s)^{-2} \exp\left[-\frac{(x-y)^2}{4s}\right] \cdot \\
&\cdot \; \left\{ \delta_{\mu\nu} \Phi^{ab}(x,y) - 2i\bar{g}s\Phi^{ac}(x,y) F^{cb}_{\mu\nu}(y) + \ldots \right\}
\end{aligned}
\tag{2.20}
$$

where

$$
\Phi(x,y) = P\exp\left[i\bar{g}\int_y^x dz_\mu A_\mu(z)\right] \tag{2.21}
$$

is the parallel transport operator along a straight line from y to x in the adjoint representation. It satisfies [15]

$$
D^{ab}_\mu \Phi^{bc}(x,y) = \frac{i\bar{g}}{2}\Phi^{ab}(x,y) F^{bc}_{\mu\nu}(y)(x_\nu - y_\nu) \tag{2.22}
$$

The actual evaluation of $T_{1,2}$ is somewhat subtle and one must carefully observe the order of the various limiting procedures involved. Let $\Omega = \Omega(\mathcal{D}_T)$ be an arbitrary operator depending on $\mathcal{D}_T$ with position–space matrix elements $\Omega^{ab}_{\mu\nu}(x,y)$. We need traces of the type

$$
\begin{aligned}
\mathrm{Tr}[\mathcal{V}\Omega] \; &\equiv \; \int d^4x d^4y \, \mathcal{V}^{ab}_{\mu\nu}(x,y) \Omega^{ba}_{\nu\mu}(y,x) \\
&= \; \int d^4x \, \phi(x) \lim_{y\to x} \Big\{ \varepsilon_{\alpha\mu\beta\nu} D^{ca}_\alpha(x) D^{cb}_\beta(y) \Omega^{ab}_{\mu\nu}(x,y) \\
&\qquad\qquad + i\bar{g} \, {}^*\!F^{ab}_{\mu\nu}(x) \Omega^{ab}_{\mu\nu}(x,y) \Big\}
\end{aligned}
\tag{2.23}
$$

Here we used (2.5) and performed an integration by parts. In accord with the arguments outlined in the introduction we dropped the surface term because the

limit $\phi(x) \to 1$ is to be performed only at the very end. Let us assume that Ω can be represented as a Laplace transform:

$$\Omega(\mathcal{D}_T) = \int_0^\infty ds\, \omega(s) e^{-s\mathcal{D}_T} \tag{2.24}$$

In our applications this will always be the case and therefore

$$\Omega^{ab}_{\mu\nu}(x,y) = \int_0^\infty ds\, \omega(s) K^{ab}_{\mu\nu}(x,y;s) \tag{2.25}$$

By inserting (2.25) with (2.20) into (2.23) and making repeated use of (2.22) one finds after a lengthy calculation

$$\mathrm{Tr}[\mathcal{V}\Omega] = -\frac{\bar{g}^2}{8\pi^2} T(G) L[\omega(s)] \int d^4x \phi(x) F^a_{\mu\nu}\, {}^*\!F^a_{\mu\nu} \;+ \dots \tag{2.26}$$

The functional $L[\omega]$ is defined in terms of a coincidence limit $z \equiv x - y \to 0$:

$$L[\omega] = \lim_{z\to 0} \frac{z^2}{4} \int_0^\infty \frac{ds}{s^2}\, \omega(s) \exp\left(-\frac{z^2}{4s}\right) \tag{2.27}$$

We observe that if $\omega(s)$ vanishes sufficiently fast for $s \to 0$ the integral exists without the exponential damping factor and we get $L[\omega] = 0$ immediately. The normalization of L is such that $L[\omega] = 1$ for a constant function $\omega = 1$. Likewise one obtains for traces involving the projector $P_L = -(D \otimes D)\mathcal{D}_T^{-1}$:

$$\mathrm{Tr}[\mathcal{V}P_L\Omega] = \frac{\bar{g}^2}{32\pi^2} T(G) L[\tilde{\omega}(s)/s] \int d^4x \phi(x) F^a_{\mu\nu}\, {}^*\!F^a_{\mu\nu} \;+ \dots \tag{2.28}$$

Since P_L gives rise to an additional factor of $\mathcal{D}_T^{-1}$ one defines $\tilde{\omega}$ by $\mathcal{D}_T^{-1}\Omega(\mathcal{D}_T) = \int_0^\infty ds\, \tilde{\omega}(s) K(s)$. It is easily expressed in terms of the Laplace transform of Ω:

$$\tilde{\omega}(s) = s \int_0^1 du\, \omega(su) \tag{2.29}$$

The trace (2.26) is entirely due to the second term (proportional to $sF^{cd}_{\mu\nu}$) in the curly brackets of eq.(2.20). The projected trace (2.28) receives contributions only from the first term proportional to $\delta_{\mu\nu}$. This explains the additional factor of $1/s$ in the argument of $L[\tilde{\omega}(s)/s]$ in (2.28).

For the computation of T_1 it is useful to define the dimensionless function σ_1 by

$$\left[y + R^{(0)}(y)\right]^{-1} = \int_0^\infty ds\, \sigma_1(s) e^{-sy} \tag{2.30}$$

For a momentum independent (mass–type) cutoff[7] $R^{(0)}(y) = 1$, say, it reads $\sigma_1(s) = \exp(-s)$, and for the exponential cutoff [12, 5]

$$R^{(0)}(y) = y\,[e^y - 1]^{-1} \tag{2.31}$$

it is a step function: $\sigma_1(s) = \theta(1-s)$.

After these preparations we are now ready to write down the relevant term of T_1 in (2.19) as a functional of $\sigma_1(s)$:

$$T_1 = k^2 \left\{ j_1(k^2) + \frac{1}{4}(1-\alpha) j_1^{(\alpha)}(k^2) \right\} \frac{\bar{g}^2 T(G)}{4\pi^2} \int d^4x\, \phi(x) F^a_{\mu\nu} {}^*\!F^a_{\mu\nu} \tag{2.32}$$

Here

$$j_1(k^2) \equiv -L\left[\frac{d}{dk^2}\sigma_1(k^2 s)\right] \quad , \quad j_1^{(\alpha)}(k^2) \equiv -L\left[\frac{d}{dk^2}\sigma_1^{(\alpha)}(k^2 s)\right] \tag{2.33}$$

with

$$\sigma_1^{(\alpha)}(k^2 s) \equiv \int_0^1 du\, \sigma_1(k^2 s u) \tag{2.34}$$

Let us investigate the properties of the function

$$j_1(k^2) = -\lim_{z\to 0} \frac{z^2}{4} \int_0^\infty \frac{ds}{s} \exp\left(-\frac{z^2}{4s}\right) \sigma_1'(k^2 s) \tag{2.35}$$

(The prime denotes the derivative with respect to the argument.) In solving the evolution equation (2.18) we shall encounter integrals of the form

$$I = \int_{k_0^2}^\infty dk^2\, j_1(k^2)\varphi(k^2) \tag{2.36}$$

where $k_0 > 0$ is a constant and $\varphi(k^2)$ is a smooth test function which does not necessarily vanish at infinity. In our application the point–separation $z \equiv x - y \neq 0$ plays the rôle of an UV cutoff. It can be removed only after the k^2–integration has been performed. Hence (2.36) should be interpreted as

$$\begin{aligned} I &= -\lim_{z\to 0} \frac{z^2}{4} \int_0^\infty \frac{ds}{s} \exp\left(-\frac{z^2}{4s}\right) \int_{k_0^2}^\infty dk^2\, \varphi(k^2)\sigma_1'(k^2 s) \\ &= -\lim_{z\to 0} \frac{z^2}{4} \int_0^\infty \frac{ds}{s} \exp\left(-\frac{1}{s}\right) \int_{k_0^2}^\infty dk^2\, \varphi(k^2)\sigma_1'(\tfrac{1}{4}k^2 z^2 s) \end{aligned} \tag{2.37}$$

[7] Though this cutoff does not satisfy $R^{(0)}(\infty) = 0$ it may be used if it does not cause UV divergences [16].

where we rescaled $s \longrightarrow \frac{1}{4}z^2 s$ in the second line. Setting $p^2 \equiv \frac{1}{4}k^2 z^2 s$ one obtains

$$I = -\int_0^\infty \frac{ds}{s^2} \exp\left(-\frac{1}{s}\right) \lim_{z\to 0} \int_{\frac{1}{4}z^2 k_0^2 s}^\infty dp^2\, \varphi\left(\frac{4p^2}{z^2 s}\right) \sigma_1'(p^2) \tag{2.38}$$

Because the s–integral is well convergent for both $s \to \infty$ and $s \to 0$ it commutes with the limit $z \to 0$. For σ_1 regular, the p^2–integral becomes in this limit

$$\varphi(\infty) \int_0^\infty dp^2\, \sigma_1'(p^2) = \varphi(\infty)\, [\sigma_1(\infty) - \sigma_1(0)] \tag{2.39}$$

Because (2.30) and $R^{(0)}(0) = 1$ imply that $\int_0^\infty ds \sigma_1(s) = 1$ we have $\sigma_1(\infty) = 0$. It is also easy to see that $\sigma_1(0) = 1$ which follows from

$$1 = \lim_{y\to\infty} \frac{y}{y + R^{(0)}(y)} = -\lim_{y\to\infty} \int_0^\infty ds\, \sigma_1(s) \frac{d}{ds} e^{-sy} \tag{2.40}$$

after an integration by parts. Inserting (2.39) into (2.38) leads to the remarkable result that

$$\int_{k_0^2}^\infty dk^2\, j_1(k^2) \varphi(k^2) = \varphi(\infty) \tag{2.41}$$

Thus, with the understanding that the coincidence limit is performed after the integration, we find that the "function" j_1 actually is a distribution which has the character of a δ–peak located at infinity. Though this behavior might seem strange at first sight it is precisely what one would expect on physical grounds. As we shall see in detail later on, the renormalization of the topological charge by gauge boson loops is a phenomenon which is very similar to the chiral anomaly of fermions. In either case the essential physics is contained in (carefully regularized) short distance singularities of operator products.

The analysis for $j_1^{(\alpha)}(k^2)$ proceeds along the same lines with σ_1 replaced by $\sigma_1^{(\alpha)}$ and one finds

$$\int_{k_0^2}^\infty dk^2\, j_1^{(\alpha)}(k^2) \varphi(k^2) = \varphi(\infty) \tag{2.42}$$

One of the interesting properties of the integrals (2.41) and (2.42) is that they do not depend on the precise form of the cutoff $R^{(0)}(y)$: they describe *universal* properties of the renormalization group flow. Coming now to the second piece on the RHS of the evolution equation, T_2, this is not the case any longer. T_2 is proportional to the anomalous dimension η_F and contains the higher order corrections therefore. It is most easily calculated in terms of the Laplace transform σ_2 defined by

$$R^{(0)}(y)[y + R^{(0)}(y)]^{-2} = \int_0^\infty ds\, \sigma_2(s) e^{-sy} \tag{2.43}$$

One obtains

$$T_2 = -k^2 \left\{ j_2(k^2) + \frac{1}{4}(1-\alpha) j_2^{(\alpha)}(k^2) \right\} \eta_F(k) \frac{\bar{g}^2 T(G)}{8\pi^2} \int d^4x\, \phi(x) F^a_{\mu\nu} {}^*\!F^a_{\mu\nu} \tag{2.44}$$

with

$$j_2(k^2) \equiv k^{-2} L\left[\sigma_2(k^2 s)\right] \quad , \quad j_2^{(\alpha)}(k^2) \equiv k^{-2} L\left[\sigma_2^{(\alpha)}(k^2 s)\right] \tag{2.45}$$

and

$$\sigma_2^{(\alpha)}(k^2 s) \equiv \int_0^1 du\, \sigma_2(k^2 s u) \tag{2.46}$$

By an analysis similar to the one above one can derive that for $0 < k_0 < \infty$

$$\int_{k_0^2}^{\infty} dk^2\, j_2(k^2) \varphi(k^2) = \int_{k_0^2}^{\infty} dk^2\, j_2^{(\alpha)}(k^2) \varphi(k^2) = \xi\, \varphi(\infty) \tag{2.47}$$

The constant ξ is given by

$$\xi = \int_0^\infty \frac{ds}{s} \sigma_2(s) = \int_0^\infty dy\, R^{(0)}(y) [y + R^{(0)}(y)]^{-2} \tag{2.48}$$

We observe that j_2 and $j_2^{(\alpha)}$ too are delta–distributions with a peak at infinity, but unlike j_1 and $j_1^{(\alpha)}$ they are not universal. Their normalization ξ depends on the cutoff function $R^{(0)}$. For $R^{(0)} = 1$ one has $\xi = 1$, for instance, and the exponential cutoff (2.31) yields $\xi = \ln(2)$. This cutoff or scheme dependence of the higher order corrections is a familiar phenomenon [12]. It cancels at the level of observable quantities.

Let us now insert T_1 and T_2 from (2.32) and (2.44) into the evolution equation. Switching from k to k^2 as the evolution parameter, (2.18) becomes

$$\begin{aligned} \frac{d}{dk^2}\theta(k) \; &= \; \theta(k) Z_F(k)^{-1} \frac{\bar{g}^2}{4\pi^2} T(G) \Bigg\{ \left[j_1(k^2) + \frac{1}{4}(1-\alpha) j_1^{(\alpha)}(k^2) \right] \\ &\qquad -\frac{1}{2}\eta_F(k) \left[j_2(k^2) + \frac{1}{4}(1-\alpha) j_2^{(\alpha)}(k^2) \right] \Bigg\} \end{aligned} \tag{2.49}$$

By integrating this equation from an arbitrary $k_0^2 > 0$ to infinity and taking advantage of the δ-function nature of the j's, eqs.(2.41), (2.42) and (2.47), one arrives at

$$\theta(k_0) = \left[1 - \frac{\bar{g}^2}{4\pi^2} T(G) \left(1 + \frac{1}{4}(1-\alpha) \right) \left\{ 1 - \frac{1}{2} \xi\, \eta_F(\infty) \right\} \right] \theta(\infty) \tag{2.50}$$

This is our final result for all strictly positive scales $k_0 > 0$. The θ–parameter is renormalized relative to its bare value $\theta(\infty)$ by a finite, k–independent factor.

The function $\theta(k)$ is constant almost everywhere, but it has a finite discontinuity at infinity. When compared to "ordinary" coupling constants such as $g^2(k)$, for instance, a renormalization group trajectory of this kind is quite unusual. However, in the appendix we show in detail that this behavior is precisely the way in which the pathologies of the triangle anomaly manifest themselves in the renormalization group framework used here. The above calculation amounts to computing the renormalized vacuum expectation value of $F^a_{\mu\nu}{}^*\!F^a_{\mu\nu} \sim \partial_\mu K_\mu$ where K_μ denotes the Chern–Simons current. This calculation has many features in common with its fermionic counterpart where K_μ is replaced by the axial vector current $J^5_\mu \equiv \bar{\psi}\gamma_\mu\gamma_5\psi$. The jump of θ and of $< F^a_{\mu\nu}{}^*\!F^a_{\mu\nu} >$ corresponds to the anomaly term in $\partial_\mu J^5_\mu$. For a detailed comparison we refer to [1, 3, 17]. Similar "bosonic anomalies" are known to occur when one quantizes antisymmetric tensor fields in a gravitational background[18, 19].

In the Feynman gauge $\alpha = 1$ and with the higher order corrections neglected ($\eta_F \to 0$), our eq.(2.50) is consistent with the one–loop results of refs.[1] and [3]. In the truncation used in this paper we find additional contributions which partially sum up the effects of the higher loop orders. They are proportional to $\eta_F(\infty) \equiv \bar{g}^{-2}\beta_{g^2}(\bar{g})$ with β_{g^2} given by (2.17) in terms of the bare gauge coupling $\bar{g} \equiv g(\infty)$. This suggests that, at the level of the effective average action, the change of θ is not saturated by its one–loop value.

As for the terms proportional to $(1-\alpha)$, our result (2.50) coincides neither with [2] nor with [1]. These terms originate from the traces of the type $\mathrm{Tr}[P_L(\ldots)]$ which describe longitudinal gauge bosons circulating inside the loops. At first sight the α–dependence comes as a surprise since the background field satisfies $D_\mu F_{\mu\nu} = 0$, i.e., it is "on shell". However, as a regulating device we kept $x \neq y$ until the evolution equation was integrated. In practice a non–zero point–separation introduces a kind of virtuality similar to a nonvanishing external momentum square in the case of the usual diagrammatic calculations based upon plane waves. Thus the status of the α–dependence is the same as discussed in detail by Shifman and Vainshtein [1].

3 Infrared Renormalization

Up to now we derived and solved the evolution equation for $\theta(k)$ from infinity down to a scale k_0 which may be chosen arbitrarily low but must be kept different from zero. It is easy to convince oneself that the derivation of the previous section does not hold for the precise equality $k_0 = 0$. In fact, we are now going to show that for $k_0 \to 0$ the function $\theta(k)$ suffers from a second discontinuity [3]. The physical origin of this second jump are the zero–modes of the operator $\mathcal{D}_T$. One of the big advantages of the method employed here is that the beta–functions of the generalized couplings (g and θ here) can be determined by inserting any background field which gives a nonvanishing value to the relevant field monomials. The beta–functions do not depend on the background chosen, and we may use whatever is convenient from a computational point of view [10, 12]. The limit $k_0 \to 0$ is most conveniently investigated by inserting a self–dual field into (2.8) because this will allow us to recast the problem in a fermionic language and powerful index theorems become available. Because $F_{\mu\nu} = {}^*\!F_{\mu\nu}$ implies $D_\mu F_{\mu\nu} = 0$, the simplifications of the evolution equation made in sect.2 are still allowed, and we can rewrite (2.8) as

$$\begin{aligned} \frac{d}{dk^2}\theta(k) \int d^4x\, \phi(x) F^a_{\mu\nu} {}^*\!F^a_{\mu\nu} \ = \\ -2\theta(k) \mathrm{Tr}\left[\mathcal{V}\Big(Z_F(k)\mathcal{D}_T + R_k(\Delta)\Big)^{-2} \frac{d}{dk^2} R_k(\Delta)\right] \end{aligned} \tag{3.1}$$

For simplicity we set $\alpha = 1$ in this section, i.e., $\mathcal{D} = \mathcal{D}_T$. It is obvious from (3.1) that a zero eigenvalue of $\mathcal{D}_T$ produces a highly divergent contribution to the trace when $R_k \sim k^2 \to 0$. We shall see that this leads to the discontinuity of $\theta(k)$ mentioned above. While it is true that a field satisfying $F_{\mu\nu} = {}^*\!F_{\mu\nu}$ cannot disentangle the invariants $F_{\mu\nu}F_{\mu\nu}$ and $F_{\mu\nu}{}^*\!F_{\mu\nu}$ the function $Z_F(k)$ is continuous for $k \to 0$ and hence any nontrivial behavior for $k \to 0$ should be attributed to $\theta(k)$.

For self–dual backgrounds the technology of refs.[20] and [3] simplifies the analysis, and we start by defining four 4×4–matrices Ω_μ

$$(\Omega_\mu)_{\alpha\beta} = \begin{cases} \eta_{\alpha\beta\mu} & \text{if } \ \alpha = 1, 2, 3 \\ -\delta_{\beta\mu} & \text{if } \ \alpha = 4 \end{cases} \tag{3.2}$$

where $\eta_{\alpha\,\mu\nu}(\alpha = 1,2,3\,;\,\mu,\nu = 1,\ldots,4)$ is 't Hooft's symbol [21]. Using its well–known properties one can derive that

$$\begin{aligned} \left(\Omega_\mu \Omega_\nu^T\right)_{\alpha\beta} &= \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} + \varepsilon_{\mu\nu\alpha\beta} \\ \left(\Omega_\mu^T \Omega_\nu\right)_{\alpha\beta} &= \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} - \varepsilon_{\mu\nu\alpha\beta} \end{aligned} \tag{3.3}$$

If we set $\hat{D} \equiv \Omega_\mu D_\mu$, $\hat{D}^T \equiv \Omega_\mu^T D_\mu$ and use the condition $F_{\mu\nu} = {}^*\!F_{\mu\nu}$ then (3.3) implies

$$\begin{aligned} \left(\hat{D}^T \hat{D}\right)_{\alpha\beta} &= D^2 \delta_{\alpha\beta} \\ \left(\hat{D} \hat{D}^T\right)_{\alpha\beta} &= D^2 \delta_{\alpha\beta} - 2i\bar{g} F_{\alpha\beta} \end{aligned} \tag{3.4}$$

Thus $\mathcal{D}_T = -\hat{D}\hat{D}^T$ for self–dual fields. Using the Ω_μ's as building blocks, we introduce the following 8×8–matrices:

$$\Gamma_\mu = \begin{bmatrix} 0 & \Omega_\mu \\ -\Omega_\mu^T & 0 \end{bmatrix} \tag{3.5}$$

By virtue of the relations

$$\begin{aligned} \Omega_\mu \Omega_\nu^T + \Omega_\nu \Omega_\mu^T &= 2\delta_{\mu\nu} \\ \Omega_\mu^T \Omega_\nu + \Omega_\nu^T \Omega_\mu &= 2\delta_{\mu\nu} \end{aligned} \tag{3.6}$$

the Γ_μ's are seen to constitute an 8–dimensional representation of the 4–dimensional Clifford algebra:

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = -2\delta_{\mu\nu} \tag{3.7}$$

Note that $\Gamma_\mu^\dagger = -\Gamma_\mu$ because Γ_μ is real and antisymmetric. An important rôle will be played by the "chirality" operator

$$\Gamma_5 \equiv -\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.8}$$

It has the usual properties $\Gamma_5^2 = 1$ and $\{\Gamma_5, \Gamma_\mu\} = 0$. In order to reformulate the evolution equation in a "fermionic" language we need the Dirac operator

$$\not{D} = \Gamma_\mu D_\mu = \begin{bmatrix} 0 & \hat{D} \\ -\hat{D}^T & 0 \end{bmatrix} \tag{3.9}$$

Because $\not{D}^2$ is a block–diagonal matrix, we find the useful relation

$$\mathrm{tr}_8 \left[\Gamma_5 \Gamma_\mu \not{D}\, G(\not{D}^2)\right] = -\mathrm{tr}_4 \left[\Omega_\mu \hat{D}^T G(-\hat{D}\hat{D}^T) - \Omega_\mu^T \hat{D}\, G(-\hat{D}^T\hat{D})\right] \tag{3.10}$$

Here G is an arbitrary function and tr_4 and tr_8 denote the traces with respect to the 4×4 and the 8×8 matrix structures, respectively. From the identity

$$\left(\Omega_\mu \Omega_\nu^T - \Omega_\mu^T \Omega_\nu\right)_{\alpha\beta} = 2\varepsilon_{\mu\nu\alpha\beta} \tag{3.11}$$

one obtains the "spinor" representation of the operator $\mathcal{V}$:

$$\mathcal{V}_{\alpha\beta} = \frac{1}{2}\,(\partial_\mu \phi) \left[\Omega_\mu \hat{D}^T - \Omega_\mu^T \hat{D}\right]_{\alpha\beta} \tag{3.12}$$

The evolution equation contains a trace of the form

$$\begin{aligned} \mathrm{Tr}[\mathcal{V} G(\mathcal{D}_T)] &= -\frac{1}{2}\int d^4x\, \phi(x) \mathrm{tr}_c \mathrm{tr}_4 \partial_\mu < x| \left\{\Omega_\mu \hat{D}^T - \Omega_\mu^T \hat{D}\right\} G(-\hat{D}\hat{D}^T)|x> \\ &= -\frac{1}{2}\int d^4x\, \phi(x) \mathrm{tr}_c \mathrm{tr}_4 \partial_\mu <x|\Omega_\mu \hat{D}^T G(-\hat{D}\hat{D}^T) - \Omega_\mu^T \hat{D} G(-\hat{D}^T\hat{D})|x> - \Delta T \end{aligned} \tag{3.13}$$

(tr_c denotes the trace in color space.) As in sect.2 we performed an integration by parts and assumed that $\phi(x)$ falls off sufficiently fast so that there are no surface terms. In the last line of (3.13) we added and subtracted the same terms, i.e., ΔT is given by [3]

$$\Delta T = \frac{1}{2}\int d^4x\, \phi(x) \mathrm{tr}_c \mathrm{tr}_4 \partial_\mu <x|\Omega_\mu^T \hat{D} \left\{G(-\hat{D}^T\hat{D}) - G(-\hat{D}\hat{D}^T)\right\}|x> \tag{3.14}$$

Provided G is chosen in such a way that the trace actually exists one can use the method of section 2 together with the selfduality condition to show that ΔT does not contribute to the $\phi F_{\mu\nu} {}^*\!F_{\mu\nu}$–term and can be neglected therefore. The remaining terms in (3.13) have the structure of (3.10). Hence the whole trace can

be rewritten in the language of 8–component spinor matrices:

$$\begin{aligned}\mathrm{Tr}[\mathcal{V}G(\mathcal{D}_T)] &= \frac{1}{2}\int d^4x\,\phi(x)\,\mathrm{tr}_c\mathrm{tr}_8\partial_\mu\!<\!x|\Gamma_5\,\Gamma_\mu\,\not{D}\,G(\not{D}^2)|x\!> \\ &= \int d^4x\,\phi(x)\,\mathrm{tr}_c\mathrm{tr}_8\!<\!x|\Gamma_5\,\not{D}^2\,G(\not{D}^2)|x\!>\end{aligned} \tag{3.15}$$

By using (3.15) in (3.1) we arrive at the desired representation of the evolution equation:

$$\begin{aligned}\frac{d}{dk^2}\theta(k)F^a_{\mu\nu}(x)^*\!F^a_{\mu\nu}(x) = -2\theta(k)\mathrm{tr}_c\mathrm{tr}_8 <\!x|\Gamma_5\not{D}^2\Big[Z_F(k)\not{D}^2+Z_F(k)k^2R^{(0)}(\not{D}^2/k^2)\Big]^{-2} \\ \cdot\frac{d}{dk^2}\Big\{Z_F(k)k^2R^{(0)}(\not{D}^2/k^2)\Big\}\,|x\!>\end{aligned} \tag{3.16}$$

Let us pause here for a moment and recall the Atiyah–Singer index theorem for the operator $\not{D}$ [22, 3]. We assume that spacetime is a large 4–sphere. Hence the spectrum is discrete and for a given background A_μ there are $n_+[A]\,(n_-[A])$ zero modes $\psi_+\,(\psi_-)$ of chirality $+1\,(-1)$. We adopt the usual definitions $\psi_\pm = P_\pm\psi_\pm$ with the projectors $P_\pm = \frac{1}{2}(1\pm\Gamma_5)$. One has for all $t>0$

$$n_+[A]-n_-[A] = \mathrm{Tr}\left[\Gamma_5\exp\left(-t\not{D}^2\right)\right] \tag{3.17}$$

because by a standard argument [23] the non–zero modes cancel in the trace. By inserting the heat–kernel expansion for $\not{D}^2$ and letting $t\to 0$ one easily arrives at the index theorem

$$\begin{aligned}n_+[A]-n_-[A] &= 4T(G)Q \\ &= T(G)\frac{\bar{g}^2}{8\pi^2}\int d^4x\,F^a_{\mu\nu}{}^*\!F^a_{\mu\nu}\end{aligned} \tag{3.18}$$

The prefactor $4T(G)$ of the topological charge arises since we are dealing with "fermions" in the adjoint representation and because we employ a non–standard representation of the Clifford algebra.[8] The solutions to the zero–mode equation $\not{D}\psi_\pm = 0$ have the form $\psi_+ = (\phi_+,0)$ and $\psi_- = (0,\phi_-)$ where ϕ_+ and ϕ_- satisfy

[8] It enters the heat–kernel computation of the index via the identity $\mathrm{tr}_8[\Gamma_5\Gamma_\mu\Gamma_\nu\Gamma_\rho\Gamma_\sigma] = -8\varepsilon_{\mu\nu\rho\sigma}$.

$\hat{D}^T\phi_+ = 0$ and $\hat{D}\phi_- = 0$, respectively. Multiplying by $\hat{D}$ and $\hat{D}^T$ from the left we see that $\mathcal{D}_T\phi_+ = 0$ and $D^2\phi_- = 0$. Because D^2 has no zero–modes one has $n_- = 0$ so that $n_+ = 4T(G)Q > 0$ is the number of zero–modes of $\mathcal{D}_T$.[9]

Equipped with the index theorem it is easy to analyze the evolution equation (3.16). Since we already know from section 2 that $\theta(k)$ is constant even for k close to (but different from) zero, it is sufficient to integrate (3.16) from zero up to a small $k_0^2 > 0$. Because $Z_F(k)$ is continuous for $k \to 0$ we may replace $Z_F(k)$ by $Z_F(0)$ in (3.16). Thus

$$[\theta(k_0) - \theta(0)]F_{\mu\nu}(x)\tilde{F}^a_{\mu\nu}(x) = -2 < x|\mathrm{tr}_c\mathrm{tr}_8\Gamma_5 Y(\not{D}^2)|x > \tag{3.19}$$

with

$$Y(\lambda) \equiv -\lambda Z_F(0)^{-1}\int_0^{k_0^2} dk^2\,\theta(k)\frac{d}{dk^2}\left[\lambda + k^2 R^{(0)}(\lambda/k^2)\right]^{-1} \tag{3.20}$$

Seen as a function of the real parameter λ, Y has a smooth limit for $\lambda \to 0$. Writing

$$Y(\lambda) = -Z_F(0)^{-1}\int_0^{k_0^2/\lambda} dy\,\theta\left(\lambda^{\frac{1}{2}}y^{\frac{1}{2}}\right)\frac{d}{dy}\left[1 + yR^{(0)}(1/y)\right]^{-1} \tag{3.21}$$

we observe that the constant factor $\theta(0)$ emerges in the limit $\lambda \to 0$, and that y is integrated from zero to infinity. One obtains the $R^{(0)}$–independent limit

$$Y(0) = Z_F(0)^{-1}\theta(0) \tag{3.22}$$

We determine the discontinuity $\theta(k_0) - \theta(0)$ by integrating (3.19) over x and applying the index theorem. The RHS of (3.19) becomes $-2\mathrm{Tr}[\Gamma_5 Y(\not{D}^2)]$, and because the non–zero modes of $\not{D}^2$ with positive and negative chirality are always paired this equals $-2Y(0)\mathrm{Tr}[\Gamma_5]$. A regularized version of $\mathrm{Tr}[\Gamma_5]$ is provided by (3.17) so that we may replace $\mathrm{Tr}[\Gamma_5]$ by $4T(G)Q$. Putting everything together we arrive at the final answer for the jump of $\theta(k)$ near $k = 0$:

$$\theta(0) = \left[1 - \frac{1}{4\pi^2}T(G)\bar{g}^2 Z_F(0)^{-1}\right]^{-1}\theta(k_0) \tag{3.23}$$

Recall that $\bar{g}^2 Z_F(0)^{-1} = g^2(0)$ is the running gauge coupling at zero momentum and $\bar{g}^2 \equiv g^2(\infty)$ is the bare one. We can combine (3.23) with (2.50) for $\alpha = 1$ and express the renormalized θ–parameter $\theta(0)$ in terms of the bare parameter $\theta(\infty)$:

[9] There are no solutions of $\mathcal{D}_T\phi_+ = 0$ with $\hat{D}^T\phi_+ \neq 0$ because there exists a positive–definite inner product with respect to which $\hat{D}^T$ is the adjoint of $-\hat{D}$.

$$\theta(0) = \left[1 - \frac{T(G)}{4\pi^2} g^2(0)\right]^{-1} \left[1 - \frac{T(G)}{4\pi^2} g^2(\infty) \left\{1 - \frac{1}{2}\xi\, \eta_F(\infty)\right\}\right] \theta(\infty) \qquad (3.24)$$

This is our main result. It shows that if one understands the "renormalization of the θ–parameter" in the sense of performing the limit $\phi(x) \to 1$ *after* the theory has been quantized, i.e., after the evolution equation has been solved, then there is indeed a (finite) difference between the bare and the renormalized θ–parameter.

The use of an exact renormalization group equation with the truncation (2.1) amounts to a renormalization group improved one–loop calculation. Let us switch off for a moment the corrections which go beyond a standard one–loop calculation of $\theta(0)$. In this case there is no running of Z_F, i.e., $g^2(0) = g^2(\infty)$, and the term in (3.24) proportional to η_F is absent. We see that in this case $\theta(0) = \theta(\infty)$ because the discontinuities at $k = \infty$ and at $k = 0$ cancel precisely and there is no net effect left. This is the compensation which was found in ref.[3] by different methods. It can be understood in close analogy with the well–known argument which relates the Atiyah–Singer index theorem to the anomaly equation

$$\partial_\mu < \bar{\psi}\gamma_\mu\gamma_5\psi >= 2m < \bar{\psi}\gamma_5\psi > + \frac{\bar{g}^2}{16\pi^2} F^a_{\mu\nu} {}^*\!F^a_{\mu\nu} \qquad (3.25)$$

If one integrates this equation over a compact spacetime the LHS vanishes and the second term on the RHS yields twice the topological charge. The first term on the RHS equals $-2\mathrm{Tr}[\gamma_5 m(\not{D} + m)^{-1}]$ and becomes $-2(n_+ - n_-)$ in the limit $m \to 0$. Hence $(n_+ - n_-) - Q = 0$. From our discussion of fermions in the appendix it is clear that the compensation of the jumps at $k = 0$ and $k = \infty$ is completely analogous to the compensation of $(n_+ - n_-)$ and Q. The piece $(n_+ - n_-)$ coming from the "soft" operator corresponds to the jump at $k \to 0$ and the "hard" contribution from $F_{\mu\nu} {}^*\!F_{\mu\nu}$ is related to the jump at infinity.

The calculation in this paper goes beyond a one–loop computation in that it retains the running of a second coupling, $g^2(k)$. At this level of accuracy we find clear evidence for a nonvanishing renormalization of the θ–parameter. Though the non–trivial running occurs only in the extreme ultraviolet and infrared, the discontinuities of $\theta(k)$ triggered there do not compensate any longer.

4 Conclusion

In this paper we considered the topological term $\int d^4x F^a_{\mu\nu} {}^*\!F^a_{\mu\nu}$ as the limit of the non–topological interaction $\int d^4x \phi(x) F^a_{\mu\nu} {}^*\!F^a_{\mu\nu}$. We saw that if the limit $\phi(x) \to 1$ is taken *before* the renormalization group evolution, or in other words, before the quantization, then the θ–parameter is not renormalized. This is in accord with the expectations based upon the interpretation of θ as the quasi–momentum related to the "Bloch waves" of the Yang–Mills vacuum. We have also seen that if one performs the limit *after* the evolution then a nontrivial renormalization of θ occurs. We were mostly concerned with pure Yang–Mills theory where θ is renormalized multiplicatively by a finite factor. If one adds fermions (see the appendix) then there is an additional finite shift of θ. We investigated the beta–function which describes the running of $\theta(k)$. Nontrivial effects are confined to the extreme ultraviolet region where the anomaly of the Chern–Simons current gives rise to a finite discontinuity, and to the extreme infrared where the zero modes of the inverse gauge boson propagator trigger another discontinuity. At the one–loop level the two discontinuities cancel. In our more refined calculation which keeps track of the running of both $\theta(k)$ and $g(k)$, the cancellation is incomplete and the renormalized quantity $\theta(0)$ differs from the bare value $\theta(\infty)$. The basic mechanism which spoils the compensation is that the two jumps of $\theta(k)$ occur at very different scales and involve the running gauge coupling $g(k)$ at different scales therefore.

It is one of the virtues of our renormalization group approach that it allows for a clear separation of these two regimes. This is particularly important if one thinks of realistic applications to QCD, for instance. The running of θ in the UV can be reliably calculated with truncations such as the one used in this paper. The derivation of the discontinuity at $k = 0$ rests on much less solid ground. As a first step to extend the validity towards the infrared, one could use the more general truncations on which our investigation of the gluon condensation [10] was based.

The discontinuous evolution of $\theta(k)$ is closely related to a similar phenomenon in pure 3–dimensional Chern–Simons theory. In ref.[16] we showed that the well–known shift of the Chern–Simons parameter [24] is also due to a renormalization group trajectory with a discontinuity at $k = 0$. In view of the discussion in ref.[17]

this similarity is quite natural.

Remaining is the question of what is the "correct" way of treating the topological term. Is the limit $\phi(x) \to 1$ to be taken before or after the evolution? The answer is that it depends on the physical situation. If the term S_{top} of (1.1) is part of the bare action then there is certainly no reason to artificially introduce the ϕ–field, and θ is not renormalized therefore. If, however, the topological term arises from an interaction term $\phi(x) F^a_{\mu\nu}\, {}^*\!F^a_{\mu\nu}$ because some pseudoscalar $\phi(x)$ acquires an x–independent vacuum expectation value, then the second alternative applies and θ can be renormalized.

It is quite tempting to speculate that renormalization effects of θ might provide a solution to the strong CP–problem, i.e., that they explain why the θ–angle observed in nature is extremely small or zero while a value of order unity would seem much more natural. In such a scenario one would have to show that for any bare parameter $\theta(\infty)$ the renormalization group trajectory ends at a $\theta(0)$ which is (close to) zero. Our results suggest that the effect of the gluons is much more important than that of the quarks in this respect. If we assume that they are all massive, they shift $\theta(\infty)$ in the ultraviolet, but they play no role in the infrared. Also the UV–effects by the gluons are of a perturbative nature and not very important probably. However, the zero modes of the inverse gluon propagator could have a significant impact on $\theta(0)$. They act in the strong coupling regime at a large value of g^2. In fact, if we naively set $g^2(0) = \infty$ in (3.24) we find that $\theta(0) = 0$ for all bare parameters $\theta(\infty)$! Clearly it is premature to take this result too seriously since the truncation we used is by far too simple to allow for a realistic description of QCD at small momenta. Nevertheless our result indicates that such a scenario is possible in principle, and that it is worthwhile to study this mechanism with improved approximations. It is interesting that recent lattice investigations [25] and low dimensional toy models [26] also seem to support the idea that the strong CP problem could be solved within the standard model itself.

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Appendix

In this appendix we study the influence of fermions on the renormalization of the topological charge. We couple one flavor of Dirac fermions (in the fundamental representation of G) to A_μ^a, and we assume that there is also a coupling of the fermions to the external field $\phi(x)$. We investigate both a pseudovector coupling $(\partial_\mu\phi)\bar{\psi}\gamma_\mu\gamma_5\psi$ and a pseudoscalar coupling $\phi\bar{\psi}\gamma_5\psi$. The pseudovector case is the fermionic analogue of the purely bosonic effects studied in the main body of this paper. It is closely related to the standard chiral anomaly, but we include its discussion here because it is quite interesting to see its similarities and differences to the "bosonic anomaly" of section 2. The pseudoscalar coupling, on the other hand, has strikingly different properties: it leads to a smooth running of $\theta(k)$ at all scales k.

Starting with the pseudovector coupling we generalize the truncation by adding the following term to the Γ_k of eq.(2.1):

$$\Delta\Gamma_k[A,\psi,\bar{\psi};\phi] = \int d^4x \left\{ Z_\psi(k)\,\bar{\psi}\,[\not{D} + m(k)]\,\psi - i\partial_\mu\,\phi(x)\bar{\psi}\gamma_\mu\gamma_5\psi \right\} \tag{A.1}$$

We determine the scale dependence of the induced interaction $\sim \phi F_{\mu\nu}{}^*\!F_{\mu\nu}$ by solving the evolution equation for the coupled gauge field/fermion system. Its general form can be found in ref.[27]. Here the situation simplifies because in order to determine $\theta(k)$ and $Z_F(k)$ backgrounds of the type $A, \bar{A} \neq 0$, $\psi = 0 = \bar{\psi}$ are sufficient. In this case the RHS of the evolution equation is simply the sum of the two traces which are present in (1.2) plus a similar term involving the fermion–fermion submatrix of $(\Gamma_k + \Delta\Gamma_k)^{(2)}$. Hence the running of θ is governed by

$$\begin{aligned} i\frac{\bar{g}^2}{32\pi^2}k\frac{d}{dk}\theta(k)\int d^4x\,\phi(x)F_{\mu\nu}^a{}^*\!F_{\mu\nu}^a \;=& \\ -\mathrm{Tr}\left\{\left[(\Gamma_k+\Delta\Gamma_k)^{(2)}_{\bar{\psi}\psi}+R_k\right]^{-1}k\frac{d}{dk}R_k\right\}&+\ldots \end{aligned} \tag{A.2}$$

The dots represent the contributions of the gauge field and of the ghosts which we have evaluated already. A cutoff appropriate for Dirac fermions is

$$R_k = Z_\psi(k)kR^{(0)}\left(-\not{D}^2/k^2\right) \tag{A.3}$$

Expanding the RHS of (A.2) to first order in ϕ one obtains

$$\begin{aligned}
\frac{\bar{g}^2}{32\pi^2}\frac{d}{dk^2}\theta(k)F^a_{\mu\nu}(x)\tilde{F}^a_{\mu\nu}(x) &= -2Z_\psi(k)^{-1}\mathrm{tr}\gamma_5 <x|\not{D}\left[\not{D}+\mu_k(\not{D}^2)\right]^{-2} \\
&\quad \cdot\frac{d}{dk^2}\left[kR^{(0)}(-\not{D}^2/k^2)\right]|x> +\mathcal{O}(\eta_\psi) \\
&= Z_\psi(k)^{-1}\frac{d}{dk^2}\mathrm{tr}\gamma_\mu\gamma_5\gamma_\nu \lim_{y\to x}\left[D_\mu(x)+D^\dagger_\mu(y)\right]D_\nu(x) \\
&\quad \cdot <x|\left(-\not{D}^2+\mu_k(\not{D}^2)^2\right)^{-1}|y> +\mathcal{O}(\eta_\psi)+\mathcal{O}(\partial_k m)
\end{aligned} \tag{A.4}$$

with $\mu_k(\not{D}^2)\equiv m+kR^{(0)}(-\not{D}^2/k^2)$. Here we are interested in the main features only and do not evaluate the higher order terms proportional to $\eta_\psi=-k\frac{d}{dk}\ln Z_\psi$. In the last line of (A.4) we also neglected the k–dependence of m. Since mass effects play no import rôle we set $m(k)=m=$const from now on. For dimensionless variables y and κ we introduce the Laplace transform σ_ψ by

$$\left[y+\left(\kappa+R^{(0)}(y)\right)^2\right]^{-1}=\int_0^\infty ds\,\sigma_\psi(s;\kappa)e^{-sy} \tag{A.5}$$

It satisfies $\sigma_\psi(0;\kappa)=1$ and $\sigma_\psi(\infty;\kappa)=0$. The operator appearing in (A.4) can be expressed in terms of $K(s)\equiv\exp(s\not{D}^2)$:

$$\left(-\not{D}^2+\mu_k(\not{D}^2)^2\right)^{-1}=\int_0^\infty ds\,\sigma_\psi(sk^2;m/k)K(s) \tag{A.6}$$

The heat–kernel for covariantly constant fields is well–known [28]. The terms relevant in the present context are

$$\begin{aligned}
K(x,y;s) &= (4\pi s)^{-2}\exp\left[-\frac{(x-y)^2}{4s}\right]\Phi(x,y) \\
&\quad \cdot\left\{1-\frac{1}{2}i\bar{g}sF_{\mu\nu}(y)\gamma_\mu\gamma_\nu-\frac{1}{8}\bar{g}^2s^2\left(F_{\mu\nu}(y)\gamma_\mu\gamma_\nu\right)^2+\ldots\right\}
\end{aligned} \tag{A.7}$$

Note that the parallel transport operator $\Phi(x,y)$ and $F_{\mu\nu}(y)$ are matrices in the fundamental representation. Using (A.6) with (A.7) in (A.4) one finds after some calculation

$$\frac{d}{dk^2}\theta(k)=2Z_\psi(k)^{-1}j_\psi(k^2) \tag{A.8}$$

with

$$j_\psi(k^2)\equiv -L\left[\frac{d}{dk^2}\sigma_\psi(sk^2;m/k)\right] \tag{A.9}$$

By a reasoning similar to the one following eq.(2.35) one can show that

$$\int_{k_0^2}^{\infty} dk^2 \, j_\psi(k^2)\varphi(k^2) = \varphi(\infty) \tag{A.10}$$

Thus we find the same phenomenon as in the gauge field case: j_ψ is a $R^{(0)}$–independent δ–peak at infinity. Moreover, in (A.10) all dependence on the physical fermion mass m has disappeared. From(A.8) with (A.10) we get the following result for the renormalization of θ by the fermions alone ($k_0 > 0$):

$$\theta(k_0) = \theta(\infty) - 2Z_\psi(\infty)^{-1} \tag{A.11}$$

(Usually one sets $Z_\psi(\infty) = 1$.) On the RHS of (A.8) one should add the contribution from the gauge bosons which was found in section 2. Doing this, $\theta(\infty)$ in (A.11) becomes multiplied by the square bracket in (2.50).

Next we look at the impact the zero–modes of $\not{D}$ have on $\theta(k)$. As (A.11) is valid for k_0 arbitrarily close to zero, at most they can lead to a discontinuity at $k = 0$. We integrate (A.4) from 0 to a nearby point k_0:

$$\begin{aligned} &\frac{\bar{g}^2}{32\pi^2}\left[\theta(k_0) - \theta(0)\right] \int d^4x F^a_{\mu\nu} {}^*F^a_{\mu\nu} \\ &\qquad = 2\int_0^{k_0^2} dk^2 \, Z_\psi(k)^{-1} \frac{d}{dk^2} \mathrm{Tr}\left[\gamma_5 \not{D}\left(\not{D} + m + kR^{(0)}(-\not{D}^2/k^2)\right)^{-1}\right] \\ &\qquad = -2Z_\psi(0)^{-1}\mathrm{Tr}\left[\gamma_5 \frac{\not{D}}{\not{D} + m}\right] \end{aligned} \tag{A.12}$$

Because the chiralities of its excited states are always paired, only the zero modes of $\not{D}$ contribute to the last trace in (A.12). Hence for $m \neq 0$ this trace vanishes and for $m = 0$ its value is given by the index theorem $\mathrm{Tr}[\gamma_5] \equiv n_+ - n_- = Q$. Therefore we obtain for the behavior near $k = 0$

$$\theta(0) = \begin{cases} \theta(k_0) & \text{if } \; m \neq 0 \\ \theta(k_0) + 2Z_\psi(0)^{-1} & \text{if } \; m = 0 \end{cases} \tag{A.13}$$

We see that for the massless fermions $\theta(0) = \theta(\infty) + 2[Z_\psi(0)^{-1} - Z_\psi(\infty)^{-1}]$. Only if one neglects the running of Z_ψ the two discontinuities cancel. For a Dirac fermion there is no general reason for $m(k = 0)$ to vanish , and contrary to the situation with the gauge boson the jump at $k = 0$ is more the exception than the

rule. If one includes the gauge boson contribution in (A.12) the RHS of (A.13) contains an additional factor of $[1 - g^2(0)T(g)/4\pi^2]^{-1}$.

Finally let us see what happens if we introduce a "soft" coupling of $\phi(x)$ to the pseudoscalar $\bar\psi\gamma_5\psi$. We replace (A.1) by

$$\Delta\Gamma_k\left[A,\psi,\bar\psi;\phi\right] = \int d^4x\,\left\{Z_\psi(k)\bar\psi[\not{D} + m(k)]\psi + im_5(k)\phi(x)\bar\psi\gamma_5\psi\right\} \tag{A.14}$$

where m_5 is a (possibly k–dependent) coupling with the dimension of a mass. Proceeding as above and defining $\omega_5(s)$ by

$$\frac{m + kR^{(0)}(-\not{D}^2/k^2)}{-\not{D}^2 + \left[m + kR^{(0)}(-\not{D}^2/k^2)\right]^2} = \int_0^\infty ds\,\omega_5(s)K(s) \tag{A.15}$$

one obtains (up to terms proportional to $\partial_k m$)

$$\frac{\bar g}{32\pi^2}\frac{d}{dk}\theta(k)F^a_{\mu\nu}{}^*F^a_{\mu\nu} = -m_5(k)Z_\psi(k)^{-1}\frac{d}{dk}\int_0^\infty ds\,\omega_5(s) < x|\mathrm{tr}\gamma_5 K(s)|x> \tag{A.16}$$

Upon inserting the diagonal matrix element of the heat–kernel (A.7) this leads to

$$\begin{aligned}\frac{d}{dk}\theta(k) &= -m_5(k)Z_\psi(k)^{-1}\frac{d}{dk}\int_0^\infty ds\,\omega_5(s)\\ &= -m_5(k)Z_\psi(k)^{-1}\frac{d}{dk}(m+k)^{-1}\end{aligned} \tag{A.17}$$

The second line of (A.17) obtains be setting $\not{D} \to 0$ in (A.15) and using $R^{(0)}(0) = 1$. This time we find a smooth evolution of θ which is governed by the equation (leaving gauge field effects aside)

$$\frac{d}{dk}\theta(k) = \frac{m_5(k)}{Z_\psi(k)[m+k]^2} \tag{A.18}$$

It is remarkable that also this evolution is universal, i.e., independent of the shape of $R^{(0)}$. Eq.(A.18) is trivial to solve if we approximate $Z_\psi(k) \equiv 1$ and $m_5(k) \equiv m_5$:

$$\theta(k) = \theta(\infty) - \frac{m_5}{m+k} \tag{A.19}$$

For massless fermions there is (at least within this approximation) a singularity $\sim 1/k$ as k approaches zero. For $m \neq 0$ the limit is finite:

$$\theta(0) = \theta(\infty) - \frac{m_5}{m} \tag{A.20}$$

There exists a distinguished value for the coupling m_5, namely $m_5 = 2m$. For this value, the pseudovector coupling term in (A.1) is related to its pseudoscalar counterpart in (A.14) by the classical divergence equation $\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi) = 2m(\bar{\psi}\gamma_5\psi)$. From (A.20) we get $\theta(0) = \theta(\infty) - 2$ in this case, but exactly the same relation also follows from (A.11) for $Z_\psi = 1$ and $k_0 = 0$. (Recall that there is no jump at $k = 0$ for $m \neq 0$.) This is a manifestation of the "equivalence theorem" proven by Schwinger [28] long ago. Though for $m_5 = 2m$ the value of θ at $k = 0$ is the same for pseudoscalar and pseudovector couplings, we have seen that the pertinent renormalization group trajectories are quite different in the two cases.

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